Selected solution to Assignment 9

Supplementary Problems

1. The zeros of a function $F(x,y,z)=0$ may define a surface in space. Let $S=\{(x,y,z): F(x,y,z)=0\}$ where F is C^1. Suppose that $F_z\neq 0$. By Implicit Function Theorem the set S can be locally described as the graph of a function $z=\varphi(x,y)$. Suppose now $S=\{(x,y,\varphi(x,y)),\ (x,y)\in D\}$ where D is a region in the xy-plane. Derive the following surface area for S:
$$|S| = \iint_D \frac{|\nabla F|}{|F_z|}\,dA(x,y)\ .$$

   Solution. The area of the graph is given by
$$\iint_D \sqrt{1+\varphi_x^2+\varphi_y^2}\,dA(x,y)\ .$$
   By differentiating the relation $F(x,y,\varphi(x,y))=0$ in x and y to get
$$F_x + F_z\varphi_x = 0\ , \quad F_y + F_z\varphi_y = 0\ .$$
   Hence
$$1+\varphi_x^2+\varphi_y^2 = 1+\left(\frac{-F_x}{F_z}\right)^2+\left(\frac{-F_y}{F_z}\right)^2 = \frac{|\nabla F|^2}{|F_z|^2}$$
   and the desired result follows.

2. Let $(x(t),y(t)),\ t\in[a,b]$, be a curve C parametrized by t in the first and second quadrants. Rotate it around the x-axis to get a surface of revolution S.

   (a) Show that a parametrization of S is given by $(\alpha,t)\mapsto(x(t),y(t)\cos\alpha,y(t)\sin\alpha)$ $\alpha\in[0,2\pi]$, and it is regular when C is regular.

   Solution. By a direct computation,
$$\frac{\partial \mathbf{r}}{\partial\alpha} = (0,-y\sin\alpha,y\cos\alpha),$$
$$\frac{\partial \mathbf{r}}{\partial t} = (x',y'\cos\alpha,y'\sin\alpha),$$
   so
$$\frac{\partial \mathbf{r}}{\partial\alpha}\times\frac{\partial \mathbf{r}}{\partial t} = (-yy',x'y\cos\alpha,-yx'\sin\alpha)\ ,$$
   and
$$\left|\frac{\partial \mathbf{r}}{\partial\alpha}\times\frac{\partial \mathbf{r}}{\partial t}\right| = y\sqrt{x'^2+y'^2},$$
   since $y>0$.

   When C is regular, that is, $x^{'2}+y^{'2}>0$, it is clear that S is also regular.

   (b) Show that the surface area of S is given by
$$2\pi\int_C y(t)\,ds\ .$$

Solution. The parameter (α, t) lies in $D = [0, 2\pi] \times [a, b]$. By the surface area formula, the surface area of S is equal to

$$\begin{aligned} \iint_D y\sqrt{x'^2(t) + y'^2(t)}\, dA &= \int_0^{2\pi} \int_a^b y(t)\sqrt{x'^2(t) + y'^2(t)}\, dt d\alpha \\ &= 2\pi \int_a^b y(t)\sqrt{x'^2(t) + y'^2(t)}\, dt \\ &= 2\pi \int_C y\, ds\ . \end{aligned}$$

(c) When $y = \varphi(x), x \in [a, b]$, where φ is C^1, the surface area becomes

$$2\pi \int_a^b \varphi(x)\sqrt{1 + \varphi'^2(x)}\, dx\ .$$

Solution. Here the parametrization of the curve C is given by $x \mapsto (x, \varphi(x))$. Therefore, $y(x) = \varphi(x), x'(x) = 1$ and $y'(x) = \varphi'(x)$ and (c) follows from (b).

Exercises 16.5

More Parametrizations of Surfaces

33. a. Parametrization of an ellipsoid The parametrization $x = a\cos\theta,\ \ y = b\sin\theta,\ \ 0 \le \theta \le 2\pi$ gives the ellipse $(x^2/a^2) + (y^2/b^2) = 1$. Using the angles θ and ϕ in spherical coordinates, show that

$$\mathbf{r}(\theta, \phi) = (a\cos\theta\cos\phi)\mathbf{i} + (b\sin\theta\cos\phi)\mathbf{j} + (c\sin\phi)\mathbf{k}$$

is a parametrization of the ellipsoid $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$.

b. Write an integral for the surface area of the ellipsoid, but do not evaluate the integral.

Surface Area for Implicit and Explicit Forms

48. Find the area of the surface $2x^{3/2} + 2y^{3/2} - 3z = 0$ above the square R: $0 \le x \le 1, 0 \le y \le 1$, in the xy-plane.

56. Let S be the surface obtained by rotating the smooth curve $y = f(x)$, $a \le x \le b$, about the x-axis, where $f(x) \ge 0$.

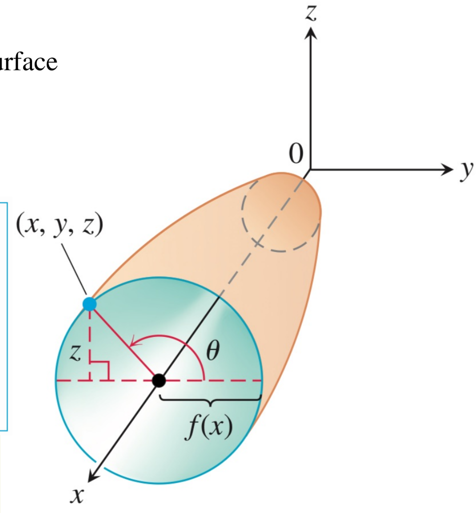
a. Show that the vector function

$$\mathbf{r}(x, \theta) = x\mathbf{i} + f(x)\cos\theta\mathbf{j} + f(x)\sin\theta\mathbf{k}$$

is a parametrization of S, where θ is the angle of rotation around the x-axis (see the accompanying figure).

b. Use Equation (4) to show that the surface area of this surface of revolution is given by

$$A = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx.$$



DEFINITION The **area** of the smooth surface

$$\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}, \qquad a \le u \le b, \quad c \le v \le d$$

is

$$A = \iint_R |\mathbf{r}_u \times \mathbf{r}_v|\,dA = \int_c^d \int_a^b |\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv. \tag{4}$$

Exercises 16.6

Surface Integrals of Scalar Functions

10. Integrate $G(x, y, z) = y + z$ over the surface of the wedge in the first octant bounded by the coordinate planes and the planes $x = 2$ and $y + z = 1$.

§16.5

Q33 (a) Substituting $\begin{cases} x = a\cos\theta\cos\phi \\ y = b\sin\theta\cos\phi \\ z = c\sin\phi \end{cases}$ into $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}$:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = \frac{(a\cos\theta\cos\phi)^2}{a^2}+\frac{(b\sin\theta\cos\phi)^2}{b^2}+\frac{(c\sin\phi)^2}{c^2}$$

$$= \cos^2\theta\cos^2\phi + \sin^2\theta\cos^2\phi + \sin^2\phi = \cos^2\phi + \sin^2\phi = 1.$$

$\therefore \begin{cases} x = a\cos\theta\cos\phi \\ y = b\sin\theta\cos\phi \\ z = c\sin\phi \end{cases}$ satisfies $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$

On the other hand: suppose (x, y, z) satisfies $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$

$\therefore \left(\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$ Hence, there exists unique $\phi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ such that

$$\begin{cases} \sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}} = \cos\phi \geq 0 \quad - (\#) \\ \frac{z}{c} = \sin\phi \Rightarrow z = c\sin\phi \; // \end{cases}$$

$(\#)$: $\frac{x^2}{a^2}+\frac{y^2}{b^2} = \cos^2\phi$. When $\phi = \pm\frac{\pi}{2}$, then $(x, y) = (0, 0) = (a\cos\theta\cos\phi, b\sin\theta\cos\phi)$.

When $\phi \neq \pm\frac{\pi}{2}$, then $\left(\frac{x}{a\cos\phi}\right)^2 + \left(\frac{y}{b\cos\phi}\right)^2 = 1$. Hence, there exists unique $\theta \in [0, 2\pi)$

such that $\begin{cases} \frac{x}{a\cos\phi} = \cos\theta \\ \frac{y}{b\cos\phi} = \sin\theta \end{cases} \Rightarrow \begin{cases} x = a\cos\theta\cos\phi \; // \\ y = b\sin\theta\cos\phi \; // \end{cases}$

(b) $\vec{r}(\theta,\phi) = a\cos\theta\cos\phi\,\vec{i} + b\sin\theta\cos\phi\,\vec{j} + c\sin\phi\,\vec{k}$.

$$\Rightarrow \begin{cases} \vec{r}_\theta(\theta,\phi) = -a\sin\theta\cos\phi\,\vec{i} + b\cos\theta\cos\phi\,\vec{j}\,; \\ \vec{r}_\phi(\theta,\phi) = -a\cos\theta\sin\phi\,\vec{i} - b\sin\theta\sin\phi\,\vec{j} + c\cos\phi\,\vec{k}. \end{cases}$$

$$\Rightarrow (\vec{r}_\theta \times \vec{r}_\phi)(\theta,\phi) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a\sin\theta\cos\phi & b\cos\theta\cos\phi & 0 \\ -a\cos\theta\sin\phi & -b\sin\theta\sin\phi & c\cos\phi \end{vmatrix}$$

$$= (bc\cos\theta\cos^2\phi)\,\vec{i} + (ac\sin\theta\cos^2\phi)\,\vec{j} + (ab\sin\phi\cos\phi)\,\vec{k}.$$

$$\Rightarrow |\vec{r}_\theta \times \vec{r}_\phi|(\theta,\phi) = (bc\cos\theta\cos^2\phi)^2 + (ac\sin\theta\cos^2\phi)^2 + (ab\sin\phi\cos\phi)^2$$

$$= b^2c^2\cos^2\theta\cos^4\phi + a^2c^2\sin^2\theta\cos^4\phi + a^2b^2\sin^2\phi\cos^2\phi$$

$\therefore$ Surface area $= \int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |(\vec{r}_\theta \times \vec{r}_\phi)(\theta,\phi)|\,d\phi\,d\theta$

$$= \int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(b^2c^2\cos^2\theta\cos^4\phi + a^2c^2\sin^2\theta\cos^4\phi + a^2b^2\sin^2\phi\cos^2\phi\right)^{\frac{1}{2}} d\phi\,d\theta \;//$$

Q48 Let $f(x,y,z) = 2x^{\frac{3}{2}} + 2y^{\frac{3}{2}} - 3z$.

$\nabla f = 3x^{\frac{1}{2}}\,\vec{i} + 3y^{\frac{1}{2}}\,\vec{j} - 3\,\vec{k}$; $\nabla f \cdot \vec{k} = -3$

$|\nabla f| = (9x + 9y + 9)^{\frac{1}{2}} = 3\sqrt{x+y+1}$; $|\nabla f \cdot \vec{k}| = 3$

$$\therefore \text{Area} = \iint_R \frac{|\nabla f|}{|\nabla f \cdot k|}\, dA = \int_0^1\int_0^1 \frac{3\sqrt{x+y+1}}{3}\, dx\, dy = \int_0^1\int_0^1 \sqrt{x+y+1}\, dx\, dy$$

$$= \int_0^1 \left[\frac{2}{3}(x+y+1)^{\frac{3}{2}}\right]_0^1 dy = \frac{2}{3}\int_0^1 \left[(y+2)^{\frac{3}{2}} - (y+1)^{\frac{3}{2}}\right] dy$$

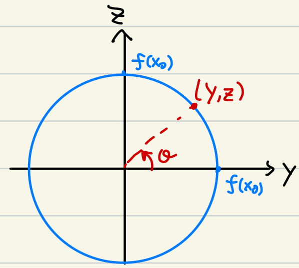
$$= \frac{2}{3}\left[\frac{2}{5}\cdot(y+2)^{\frac{5}{2}} - \frac{2}{5}(y+1)^{\frac{5}{2}}\right]_0^1$$

$$= \frac{4}{15}\left[\left(3^{\frac{5}{2}} - 2^{\frac{5}{2}}\right) - \left(2^{\frac{5}{2}} - 1\right)\right] = \frac{4}{15}\left(9\sqrt{3} - 8\sqrt{2} + 1\right) //$$

Q56 (a) For each fixed $x_0 \in [a,b]$, the cross section of S along $x = x_0$ is given by the circle centered at $(x_0, 0, 0)$ with radius $f(x_0)$, i.e.

$$S \cap \{(x_0, y, z)\} = \{(x_0, y, z) \in \mathbb{R}^3 \mid y^2 + z^2 = f(x_0)\}$$

$$= \{(x_0, f(x_0)\cos\theta, f(x_0)\sin\theta) \mid \theta \in [0, 2\pi)\}$$



where θ is also the polar angle on yz-plane.

$\therefore\ \vec{r}(x,\theta) = x\vec{i} + f(x)\cos\theta\,\vec{j} + f(x)\sin\theta\,\vec{k}$, $x \in [a,b]$, $\theta \in [0, 2\pi)$ is a parametrization of S.

(b) $\vec{r}(x,\theta) = x\vec{i} + f(x)\cos\theta\,\vec{j} + f(x)\sin\theta\,\vec{k}$;

$$\Rightarrow \begin{cases} \vec{r}_x(x,\theta) = \vec{i} + f'(x)\cos\theta\,\vec{j} + f'(x)\sin\theta\,\vec{k}; \\ \vec{r}_\theta(x,\theta) = -f(x)\sin\theta\,\vec{j} + f(x)\cos\theta\,\vec{k}; \end{cases}$$

$$\Rightarrow (\vec{r}_x \times \vec{r}_\theta)(x,\theta) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & f'(x)\cos\theta & f'(x)\sin\theta \\ 0 & -f(x)\sin\theta & f(x)\cos\theta \end{vmatrix}$$

$$= \left(f(x)f'(x)(\cos^2\theta + \sin^2\theta)\right)\vec{i} + (-f(x)\cos\theta)\vec{j} + (-f(x)\sin\theta)\,\vec{k}$$

$$= f(x)f'(x)\,\vec{i} - f(x)\cos\theta\,\vec{j} - f(x)\sin\theta\,\vec{k}$$

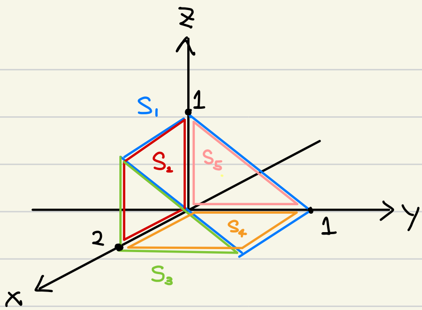
$$\therefore\ |\vec{r}_x \times \vec{r}_\theta(x,\theta)| = \left((f(x)f'(x))^2 + (f(x))^2(\cos^2\theta + \sin^2\theta)\right)^{\frac{1}{2}} = f(x)\sqrt{1 + (f'(x))^2}$$

$\therefore$ By Equation 4, $A = \int_0^{2\pi}\int_a^b \left(f(x)\sqrt{1+(f'(x))^2}\right)dx\,d\theta = 2\pi\int_a^b \left(f(x)\sqrt{1+(f'(x))^2}\right)dx$ //

§ 16.6

Q10 The surface of wedge S is divided into 5 parts

$S = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5$ as shown in the figure.



Computing the surface integral over each S_i:

- S_1: $f(x,y,z) = y+z = 1$; $G(x,y,z) = y+z = 1$.

$\nabla f(x,y,z) = \vec{j} + \vec{k}$; $|\nabla f(x,y,z)| = \sqrt{2}$; $|\nabla f(x,y,z) \cdot \vec{k}| = 1$;

$$\Rightarrow \iint_{S_1} G \, d\sigma = \int_0^1 \int_0^2 1 \cdot \sqrt{2} \, dx dy = 2\sqrt{2}$$

- S_2: $f(x,y,z) = y = 0$; $G(x,y,z) = y+z = z$.

$\nabla f(x,y,z) = \vec{j}$; $|\nabla f(x,y,z)| = 1$; $|\nabla f(x,y,z) \cdot \vec{j}| = 1$;

$$\Rightarrow \iint_{S_2} G \, d\sigma = \int_0^1 \int_0^2 z \cdot 1 \, dx dz = 2 \cdot \left[\frac{z^2}{2}\right]_0^1 = 1.$$

- S_3: $f(x,y,z) = x = 2$; $G(x,y,z) = y+z$.

$\nabla f(x,y,z) = \vec{i}$; $|\nabla f(x,y,z)| = 1$; $|\nabla f(x,y,z) \cdot \vec{i}| = 1$;

$$\Rightarrow \iint_{S_3} G \, d\sigma = \int_0^1 \int_0^{1-y} (y+z) \cdot 1 \, dz dy = \int_0^1 \left[yz + \frac{z^2}{2}\right]_0^{1-y} dy = \int_0^1 \left(y(1-y) + \frac{(1-y)^2}{2}\right) dy$$

$$= \frac{1}{2}\int_0^1 \left(2y - 2y^2 + 1 - 2y + y^2\right) dy = \frac{1}{2} \int_0^1 (-y^2+1) \, dy$$

$$= \frac{1}{2}\left[-\frac{y^3}{3} + y\right]_0^1 = \frac{1}{3}.$$

$\cdot\ S_4 : f(x,y,z) = z = 0 \ ; \ G(x,y,z) = y + z = y.$

$\nabla f(x,y,z) = \vec{k} \ ; \ |\nabla f(x,y,z)| = 1 \ ; \ |\nabla f(x,y,z)\cdot\vec{k}| = 1 \ ;$

$$\Rightarrow \iint_{S_4} G\, d\sigma = \int_0^1\int_0^2 y\cdot 1\, dx\,dy = 2\cdot\left[\frac{y^2}{2}\right]_0^1 = 1.$$

$\cdot\ S_5 : f(x,y,z) = x = 0 \ ; \ G(x,y,z) = y + z.$

$\nabla f(x,y,z) = \vec{i} \ ; \ |\nabla f(x,y,z)| = 1 \ ; \ |\nabla f(x,y,z)\cdot\vec{i}| = 1 \ ;$

$$\Rightarrow \iint_{S_5} G\, d\sigma = \int_0^1\int_0^{1-y} (y+z)\cdot 1\, dz\,dy = \int_0^1 \left[yz + \frac{z^2}{2}\right]_0^{1-y} dy = \int_0^1 \left(y(1-y) + \frac{(1-y)^2}{2}\right) dy$$

$$= \frac{1}{2}\int_0^1 \left(2y - 2y^2 + 1 - 2y + y^2\right) dy = \frac{1}{2}\int_0^1 (-y^2+1)\, dy$$

$$= \frac{1}{2}\left[-\frac{y^3}{3} + y\right]_0^1 = \frac{1}{3}.$$

$$\therefore \iint_S G\, d\sigma = \iint_{S_1} G\, d\sigma + \iint_{S_2} G\, d\sigma + \iint_{S_3} G\, d\sigma + \iint_{S_4} G\, d\sigma + \iint_{S_5} G\, d\sigma$$

$$= 2\sqrt{2} + 1 + \frac{1}{3} + 1 + \frac{1}{3} = 2\sqrt{2} + \frac{8}{3}$$ //